Stochastic resonance on a circle without excitation: Physical investigation and peak frequency formula

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In this article the existence of stochastic resonance (SR) without external force in a simplified circular system for different values of the control parameter b is considered. The average power spectra are calculated as well as the signal-to-noise ratio as a measure for stochastic resonance. It is shown that in the monostable and semistable ($b<1$ and $b=1$) cases coherent oscillations occur and SR exists. For the case $b>1$, the system is oscillatory and noise plays only a destructive role; therefore no SR occurs. The rotation number of the system is calculated and compared to the peak frequency of the power spectrum. Although the coincidence in the noisy case is not as good as that in the deterministic case, we can derive an empirical formula between the peak frequency of the power spectrum and the rotation number of the system, which is in good agreement with results of numerical simulations.

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I. INTRODUCTION

Research concerning stochastic resonance (SR) has aroused more and more interest recently in both physics and nonlinear science (see reviews [1,2]). Stochastic resonance means that the response of a nonlinear system to a periodic signal may be enhanced by an optimal amount of noise. The models investigated in the literature were mainly bistable ones with extra periodic driving forces [3–13]. Recently, some authors found that pure noise could also lead to a maximum signal-to-noise ratio (SNR) at a certain critical noise strength. This phenomenon is called stochastic resonance without excitation [14,15]. After considering the SR phenomenon in bistable systems without excitation, Hu and coworkers investigated a system that was not bistable and concluded that bistability was not a necessary ingredient for SR. In their summary paper [15], the following model was discussed:

$$dx=[x(1-x^2-y^2)+y(x-b)]dt+dw_1, \tag{1}$$

$$dy=[y(1-x^2-y^2)-x(x-b)]dt+dw_2, \tag{2}$$

where $b>0$, dw_1, and dw_2 are uncorrelated Wiener processes satisfying $\langle w_i\rangle=0$, $\langle dw_i dw_j\rangle=dt\,D\,\delta_{ij}$. In polar coordinates $x=r\cos\phi$, $y=r\sin\phi$, the deterministic part of the system can be written as

$$\dot{r}=r(1-r^2), \tag{3}$$

$$\dot{\phi}=b-r\cos\phi. \tag{4}$$

Obviously, the loop $r=1$ is the global attractor of system (3), (4). By investigating (1), (2) for the case $b=1$ in detail, Hu *et al.* pointed out that SR could occur in a system that was neither periodically forced nor bistable.

In this paper, we consider the system directly on the unit loop $r=1$ since the final behavior of system (1), (2) should be revealed by the behavior on the attractor. The equation we considered then is

$$\dot{x}=b-\sin x+D\Gamma(t), \tag{5}$$

where $b>0$, $D\geqslant 0$ represents the noise intensity, and $\Gamma(t)$ is the Gaussian white noise with zero mean and correlation given by $\langle\Gamma(t)\Gamma(t')\rangle=\delta(t-t')$. Equation (5) will be called the noisy Langevin equation (LE). The deterministic equation corresponding to system (5) is

$$\dot{x}=b-\sin x. \tag{6}$$

We plot the phase curves of system (6) for three different values of control parameter b ($b<1$, $b=1$, $b>1$) in Fig. 1. For the case $b<1$ [Fig. 1(a)], the system has a stable fixed point S_k as well as an unstable fixed point U_k in every strip $[k\pi,(k+2)\pi]$ of x, $k\in Z$. The threshold between these two points (the negative part of $b-\sin x$) is clear in the picture. If we increase the control parameter b, the threshold will decrease correspondingly and will become marginal when b is increased to the value of 1 [Fig. 1(b)]. In this case, there is only one fixed point M_k, which is actually semistable rather than monostable in every strip $[k\pi,(k+2)\pi]$ of x. This is just the corresponding case to model 2 in Ref. [15]. When the control parameter $b>1$ [Fig. 1(c)], the fixed point disappears and no threshold exists. With the guidance of these three pictures, it is convenient to investigate the influence of white noise on system (6). Precise details will be given in the later part of the article. It will be seen then that Fig. 1 is essential for explaining the physical mechanism of the occurrence of SR.

On the other hand, in dynamics, the rotation number of a system moving on a circle is an important quantity to reflect

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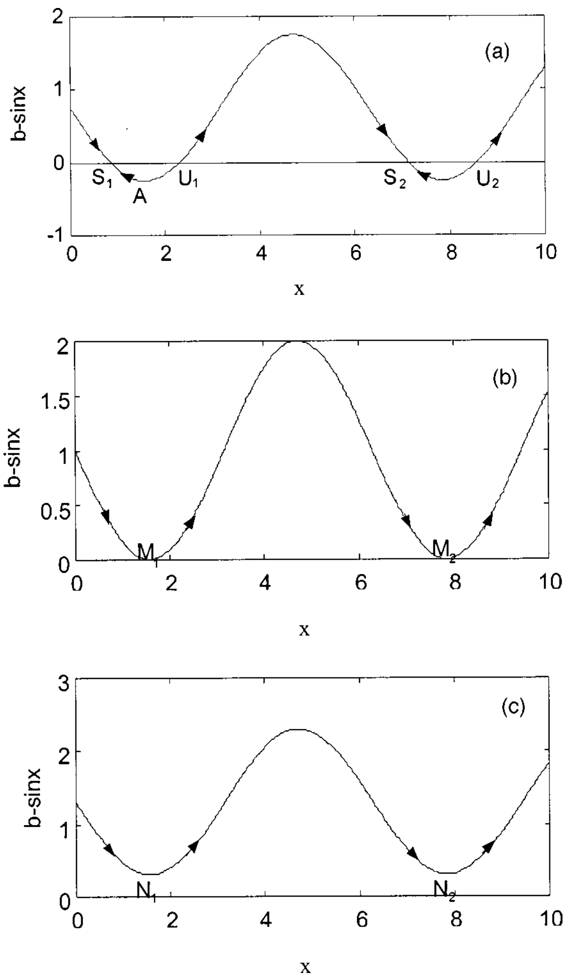

FIG. 1. $b-\sin x$ versus x for different values of control parameter b: $b<1$ (a), $b=1$ (b), $b>1$ (c).

somewhat the periodic behavior of the system. Intuitively, it scales the frequency of the rotation on the circle. The rigorous definition is

$$r(b,D)=\lim_{t\to\infty} x(t)/(2\pi t). \tag{7}$$

It is known that for a deterministic circular motion the rotation number is exactly equal to the peak frequency of the power spectrum. So it is natural to ask what will happen in a stochastic case. In this article, we will explore the relationship between these two quantities when the system is subjected to white noise.

The paper is organized as follows. In Sec. II, we elaborate the physical mechanism of the occurrence of SR in the LE, where three cases ($b<1$, $b=1$, $b>1$) are investigated. For the case $b<1$, SR is shown by computing the average power spectrum as well as the SNR. The case $b=1$ is only slightly touched since the results are the same as those in Ref. [15]. Finally, it is pointed out that for the case $b>1$ noise can only distort the coherent motion that already exists in the deterministic case and no SR exists. In Sec. III, we explore the relationships between the peak frequencies of the power spectra and the rotation numbers for the above three cases. An empirical formula is derived to calculate the peak frequencies of the power spectra.

II. PHYSICAL INVESTIGATION OF SR AND NUMERICAL SIMULATION

In the following, we will consider the influence of white noise on system (6). Here we use the power spectrum as well as the SNR, just as in Ref. [15], to describe the response of the system to the added white noise. We first pay some attention to the definition of the power spectrum.

The solution $\{x(t)\}$ to system (5) is a stationary stochastic process, the trajectory of which can be wound on the unit circle S^1 by mod(2π) because of the periodicity of $b-\sin x$ with respect to x. So we consider as usual the power spectrum of $\{\sin x(t)\}$ [equivalent to mod (2π)]. In principle, we should write $\{x(t)\}$ as $\{x(t,\zeta)\}$ since $\{x(t)\}$ is a stochastic process; here the variable ζ causes randomness. We set

$$f(\omega,\zeta)=\lim_{T\to\infty}\frac{1}{T}\int_{-T}^{T} e^{i\omega t}\sin x(t,\zeta)dt,$$

Then the power spectrum $S(\omega,\zeta)$ of the system is defined as $|f(\omega,\zeta)|^2$. However, by the ergodic theorem, $f(\omega,\zeta)$ is a deterministic function of ω, which does not depend on ζ, so we denote it as $f(\omega)$. Thus $S(\omega,\zeta)$ can be denoted as $S(\omega)$. In numerical simulations, it is necessary to take the time series sufficiently long. Equivalently, we can take adequate runs of time series and get $S(\omega)$ as their averaged power spectrum. Here we take 500 runs of independent time series $\{x_k(t)\}$ by a simple Euler forward procedure and calculate the corresponding Fourier transform $\hat{f}_k(\omega)$, which is the approximation of $f_k(\omega)$, $k=1,...,500$. The power spectrum we need is then obtained as

$$\langle S(\omega)\rangle=\sum_{k=1}^{500} |\hat{f}_k(\omega)|^2/500.$$

A. $b<1$ (monostable)

Because of the periodicity of $b-\sin x$, system (5) can be regarded as the dynamics of an overdamped particle moving on a circle S^1 driven by a constant force plus noise. Since $x(t)$ increases whenever $b-\sin x(t)>0$ and decreases whenever $b-\sin x(t)<0$ [see Fig. 1(a)], the particle starting from a certain point (say point A between the stable fixed point S_1 and the unstable fixed point U_2) will asymptotically approach S_1. So in the deterministic case the rotation number is just zero. However, the situation will be completely different when noise is included. There is a nonzero probability that the noise produces a large positive force, which helps the particle to surmount the threshold between S_1 and U_1. Instead of staying at the stable state S_1, the particle first oscillates in the neighborhood (called the attracting basin) of this stable point for some time, depending on the intensity of the noise. Then, at a certain random time, it escapes from the attracting basin of S_1 and completes a circulation after moving to S_2, which is equivalent to S_1 on the circle. The positiveness of b and the normal distribution of the noise ensure that this process is positive recurrent. Thus noise-induced rotations are manifested and coherent motion appears, which can be directly seen from the nonzero rotation number. With increase of the noise strength, such circular motion happens more and more frequently. It is easy to imagine that the

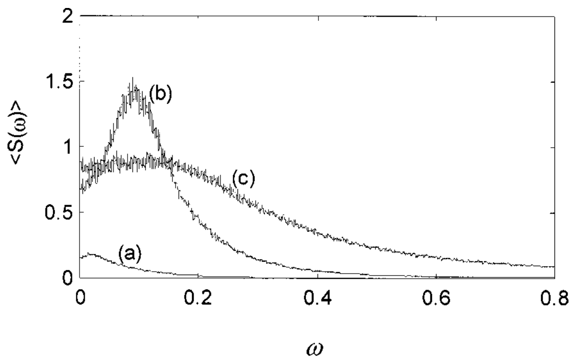

FIG. 2. The average power spectra of $x(t)$ for the case $b=0.98$ with $D=0.007$ (a), 0 (b), and 1.5 (c).

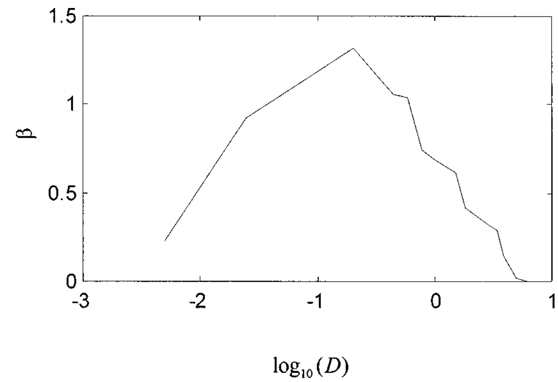

FIG. 3. The signal-to-noise ratio β against $\log_{10}(D)$ for the case $b=0.98$, SR can be seen clearly.

rotation number becomes larger and larger correspondingly, but this is not the case for coherent motion, which is destroyed if the noise intensity is too large.

As to computer simulation, we plot the power spectra $\langle S(\omega)\rangle$ of the system with respect to frequency ω for $b=0.98$ and different values of D in Fig. 2. For small noise, a low spectrum peak occurs at a small frequency [Fig. 2(a), $D=0.007$]. As the noise strength increases, the peak moves toward larger frequency; meanwhile, the height of the peak h increases. At a certain value of D (about $D\approx 0.5$), h reaches a maximum [Fig. 2(b)]. Then it decreases with further increase of noise intensity. If the strength of the noise is sufficiently large, no distinct peak in the power spectrum is observed [Fig. 2(c), $D=1.5$]. Thus the strongest coherent oscillation happens at an intermediate value of D.

To confirm the above phenomenon, we also plot the signal-to-noise ratio β versus $\log_{10}(D)$ in Fig. 3. Here we follow [15], taking $\beta=\omega_p h/W$, where ω_p is the peak frequency, and W is the width of the peak at the height of $h/\sqrt{e}$. The figure shows that the β-$\log_{10}(D)$ curve is convex. In the region of positive slope, the enhancement of the signal-to-noise ratio (SNR) manifests that the coherent motion becomes stronger as D increases. In the region with negative slope, the decrease of β informs us that the noise is gradually destroying the coherent motion as its strength increases.

The fact that the quality factor β passes through a maximum at a certain value of D indicates the existence of a best coherent motion, called a SR-like response since it resembles the usual SR phenomenon. Thus we confirm the results in [15] that SR can happen even in real monostable autonomous systems.

From the above analysis and computer simulation, we see that noise can induce circular rotations, the frequency of which can be reflected in both the rotation number of the system and the peak frequency of the power spectrum. In a deterministic case, they are just equal. How is it in a stochastic case? We think that a discussion of the relationship between these two quantities in a white noise background should be interesting. Further studies will be given in Sec. III.

B. $b=1$ (marginally stable)

The behavior of the system in this case is just the same as that of model 2 in Ref. [15]. The existence of a SR-like response for this case is well shown there. We will not discuss it further in this paper. Here we only briefly indicate the physical mechanism of SR according to Fig. 1(b). As can be seen from the picture, where S_k and U_k coincide at the point M_k and the deterministic system is marginally stable, even a slight perturbation can motivate the particle to move away from the semistable fixed point M_k ($k=1,2...$). Under white noise perturbation, the particle will oscillate near M_1 for some random time before entering the neighborhood of point M_2. So the mechanism of SR is similar to that in the case $b<1$ except that coherent motion happens much more easily since there is no threshold for the particle to overcome.

C. $b>1$

For the deterministic case, there is no equilibrium point. So without noise the particle is already rotating periodically on the unit circle in one direction (suppose it is counterclockwise); therefore, the power spectrum is discrete.

Now, if a small noise is included, most probably it can produce only small forces. So the particle can still rotate counterclockwise on the circle, occasionally interrupted by the noise. However, the power spectrum becomes continuous with more than one narrow peak and the heights of the peaks are reduced [see Fig. 4(a)]. As the value of D becomes bigger, the counterclockwise rotation is disturbed more strongly by the noise, the effect of which is especially clear near the minimum points $N_1, N_2, \ldots$ of the phase curve. In the profile of the power spectrum, only one peak with wider width and

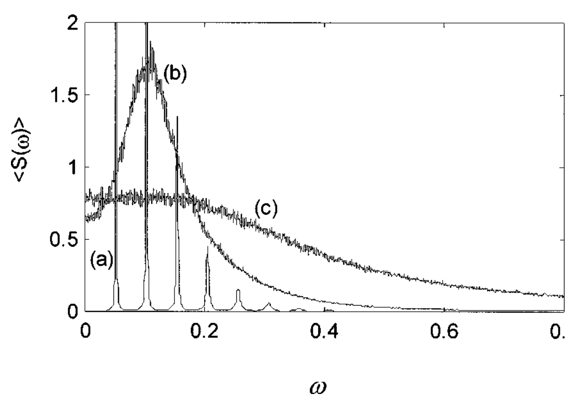

FIG. 4. The average power spectra of $x(t)$ for the cases $b=1.05$ with $D=0.005$ (a), 0.5 (b), and 1.7 (c).

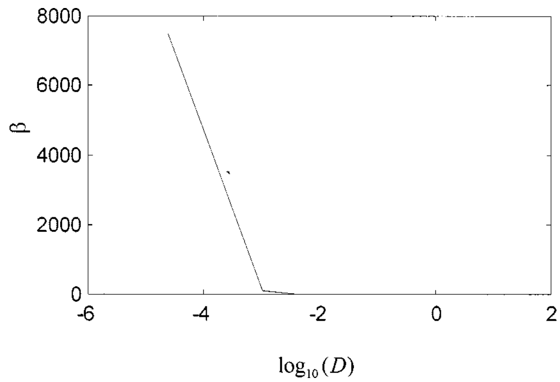

FIG. 5. The signal-to-noise ratio β against $\log_{10}(D)$ for the case $b=1.05$. No SR occurs.

larger frequency appears [Fig. 4(b)]. This indicates that the noise is gradually destroying the coherent motion displayed in the deterministic case. If the noise intensity is sufficiently large, the peak becomes ambiguous [Fig. 4(c)], which means that the system is completely dominated by the noise.

Since the influence of the noise can be seen more clearly in the SNR, we plot this quality factor β of the system against $\log_{10}(D)$ in Fig. 5. When noise is included, the width of the peak is no longer zero and the quality factor β is reduced to a finite value; in fact, it decreases to zero if the value of D is large enough. This means that no SR occurs in this case.

III. RELATIONSHIP BETWEEN PEAK FREQUENCY AND ROTATION NUMBER

Now let us explore the relationship between the peak frequency ω_p of the power spectrum and the rotation number $r(b,D)$ of the system. According to the definition [Eq. (7)], the rotation number $r(b,D)$ should depend on the trajectory in a stochastic situation. However, the system we discussed here proves to be an ergodic stationary stochastic process, so $r(b,D)$ is independent of the trajectory. In a numerical simulation, instead of taking a long time series, we can take adequate trajectories (here we take 500 runs), and calculate the average rotation number. In fact, in the Appendix, we obtain an analytic formula for the rotation number which is independent of the trajectory of system (5) (for more detail, see Ref. [16]). This formula is also used in calculating $r(b,D)$, which shows good agreement with the result from Eq. (7).

A. $b<1$

In Fig. 6(a), we plot the peak frequency ω_p of the power spectrum and the rotation number $r(b,D)$ of the system versus $\log_{10}(D)$ (solid line and dashed line, respectively). From the profiles, we can see that ω_p and $r(b,D)$ both increase monotonically with the increase of noise strength until ω_p reaches a maximum. In this region, the peak frequency ω_p is bigger than the rotation number $r(b,D)$, but if we further increase D, the rotation number $r(b,D)$ still increases while the peak frequency ω_p becomes indeterminable since the peak of the power spectrum is not obvious.

Now the question arising here is what causes such a relationship between the peak frequency ω_p and the rotation number $r(b,D)$? We focus on the value of D where the quality factor β is not too small. From the analysis for the case $b<1$ in Sec. II, we know that, if noise is included, the particle will oscillate near every identical stable fixed point S_k ($k\in Z$) of the phase curve for some time. By investigating the stochastic orbits [here we consider the time series of $\{\sin x(t)\}$] of the system, we find that, excluding wandering in the attracting basin, the remaining motion of the particle is nearly periodic (see Fig. 7). We assert that this remaining motion outside the attracting basin mainly determines the peak frequency. However, according to definition (7), the calculation of the rotation number should include the time spent oscillating near the stable fixed points. So the rotation number $r(b,D)$ is smaller than the peak frequency ω_p if the noise strength is not too large.

To make the above judgment clearer, denote T as the total time for the particle to move on the unit circle and T_0 as the time the particle moves outside the attracting basin. Then $Tr(b,D)$ counts the total number of times that the particle rotates around the circle, so $Tr(b,D)/T_0$ counts the frequency of the particle moving outside the attracting basin,

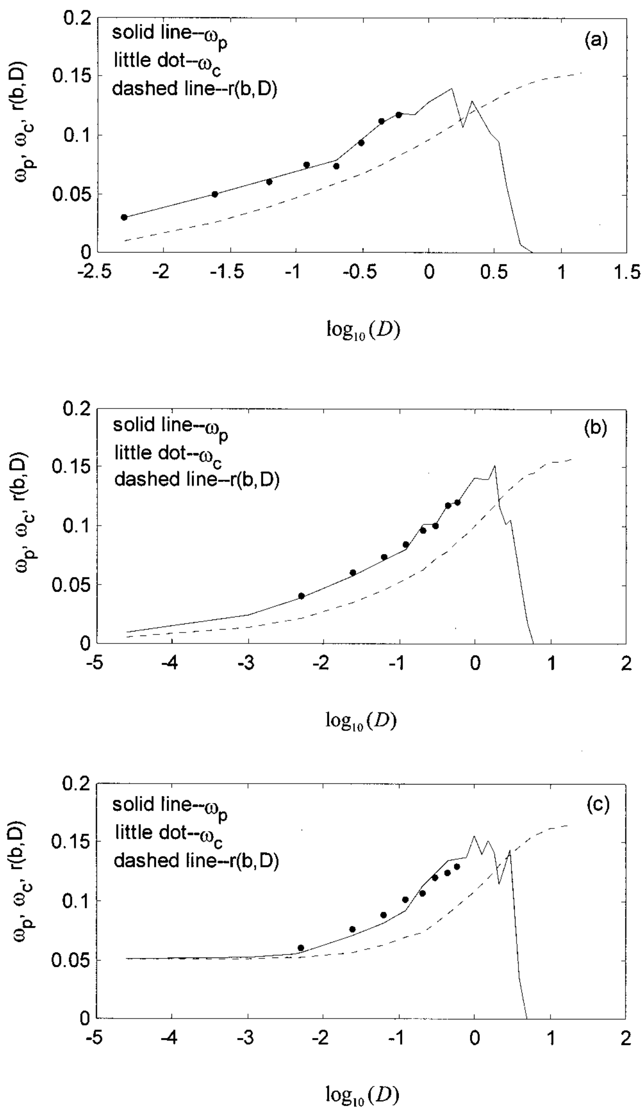

FIG. 6. The peak frequency ω_p, the calculated frequency ω_C, and the rotation number $r(b,D)$ against $\log_{10}(D)$ for different values of b: $b=0.98$ (a), $b=1$ (b), $b=1.05$ (c).

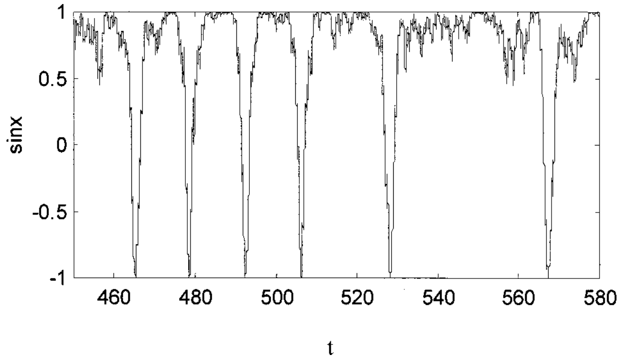

FIG. 7. The time series of $\{\sin x(t)\}$ versus t for $b=0.98$ and $D=0.4$. The motion of the particle oscillating near the stable point as well as outside the attracting basin can be clearly seen.

denoted ω_c. Comparing this frequency ω_c with the peak frequency ω_p of the power spectrum, we found them to be in good agreement. The relative error $|\omega_p-\omega_c|/\omega_p$ is about 6% [see solid line and little dots in Fig. 6(a)]. Here the sample points of D are taken between 0.1 and 0.8 with step 0.1. So we get an empirical formula to calculate the peak frequency of the power spectrum, that is,

$$\omega_p \approx \omega_c, \quad \text{i.e.,} \quad \omega_p \approx Tr(b,D)/T_0. \tag{8}$$

We call ω_c the calculated frequency of the power spectrum. According to formula (8), we have reason to say that the motion excluding the oscillations near the stable point mainly determines the peak frequency of the power spectrum.

B. $b=1$

For this case we know that if noise is included, in every circulation, the particle has to spend some random time oscillating near the semistable fixed point. So for not too large values of D the rotation number $r(b,D)$ is also smaller than the peak frequency ω_p. But, since the threshold is marginal, the time the particle spends wandering in the neighborhood of the semistable fixed point is relatively short in contrast to the case $b<1$. Thus the rotation number $r(b,D)$ and the peak frequency ω_p are more similar than that in the case $b<1$ [see solid line and dashed line in Fig. 6(b)]. After calculating the value of $Tr(b,D)/T_0$, we see that the empirical formula (8) still holds for this case [see solid line and little dots in Fig. 6(b)].

C. $b>1$

The relationship between ω_p and $r(b,D)$ in this case can be seen clearly from the solid line and dashed line in Fig. 6(c). For small values of D, the rotation number $r(b,D)$ agrees with the peak frequency ω_p well because the forces produced by the noise can seldom prevent the particle from rotating counterclockwise. If the noise intensity increases, the particle will also oscillate near the minimum points $N_1, N_2, \ldots$ of the phase curve for some random time. Thus the relationship between ω_p and $r(b,D)$ is similar to that in the cases $b<1$ and $b=1$ except that the two curves show a much better degree of similarity. Moreover, the formula (8) remains correct [see solid line and little dots in Fig. 6(c)].

IV. CONCLUSIONS

First, SR-like phenomena are manifested for the cases $b<1$ and $b=1$. From the figures, the height of the peak and the signal-to-noise ratio go through a process of first increasing and then decreasing. However, for the case $b>1$, these two quantities always decrease with increasing noise strength. From this point of view, the SR-like response does not exist for the case $b>1$. Thus we confirm the results in [15] that SR exists even in real monostable autonomous systems without periodic excitation; this is the case $b<1$. Moreover, we point out that when the control parameter $b>1$ noise can play only a negative role, and no SR exists.

Secondly, noise breaks the equality between the peak frequency ω_p and the rotation number $r(b,D)$, which holds in the deterministic case. Actually, the rotation number is smaller than the peak frequency for not too large a value of D. This is because, in the presence of noise, irregular movement in the neighborhood of stable equilibrium points ($b<1$) or of minimum points ($b>1$, $b=1$) of the phase curve occupies a certain time. Furthermore, an empirical formula is derived to show the relationship between these two quantities.

Thirdly, from the empirical formula, we can also give an explanation of the peak frequency. It is mainly determined by the motions of the particle outside the attracting basins ($b<1$) or outside the neighborhood of the minimum points ($b>1$, $b=1$) of the phase curve.

Finally, we want to point out that, although the empirical formula shows good agreement with results of numerical simulations, a mathematical demonstration is needed, but it is not easy to give it.

APPENDIX

We define the rotation number $r(b,D)$ as Eq. (7). As for the LE (5), the solution $\{x(t),t\geqslant 0\}(\mathrm{mod}\,2\pi)$ is a positive recurrent diffusion process on the unit circle and there is a unique invariant distribution. The infinitesimal generator of $\{x(t),t\geqslant 0\}$ is

$$Lf=\frac{D^2}{2}\frac{d^2}{dx^2}f(x)+\frac{d}{dx}[(b-\sin x)f(x)].$$

By the Birkhoff ergodic theorem

$$\begin{aligned} r(b,D) &= \lim_{t\to\infty}\frac{x(t)}{2\pi t} \\ &= \frac{1}{2\pi T}\lim_{T\to\infty}\left\{\int_0^T[b-\sin x(s)]ds+bw(T)\right\} \\ &= \frac{1}{2\pi}\int_0^{2\pi}(b-\sin x)\nu(x)dx, \end{aligned}$$

where $\nu(x)$ is the unique stationary density solution to the Fokker-Planck equation related to the LE system:

$$\frac{\partial \nu}{\partial t}=\frac{b^2}{2}\frac{\partial^2\nu}{\partial x^2}-\frac{\partial}{\partial x}[(b-\sin x)\nu], \quad \nu(x,t)=\nu(x+2\pi,t).$$

Let

$$w(x)=\frac{2}{D^2}\exp[f(x)]\left\{\int_0^x \exp[-f(s)]ds + \int_x^{2\pi}\exp\left(-f(s)-\frac{4\pi b}{D^2}\right)ds\right\},$$

$$\|w(x)\|=\int_0^{2\pi} w(x)dx,$$

where $f(x)=bx+\cos x-1$. Then $\nu(x)=w(x)/\|w(x)\|$, and we get

$$r(b,D)=[\exp(4\pi b/D^2)-1]/\|w(x)\|.$$

By further calculation, we can prove mathematically that $r(b,D)$ is monotonic with D and has a bounded limit when D tends to infinity.

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